

Fermionic zero modes in self-dual vortex background on a torus*

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Abstract

We study fermionic zero modes in the self-dual vortex background on an extra two-dimensional Riemann surface in 5+1 dimensions. Using the generalized Abelian Higgs model, we obtain the inner topological structure of the self-dual vortex and establish the exact self-duality equation with topological term. Then we analyze the Dirac operator on an extra torus and the effective Lagrangian of four-dimensional fermions with the self-dual vortex background. Solving the Dirac equation, the fermionic zero modes on a torus with the self-dual vortex background in two simple cases are obtained.

PACS numbers: 11.10.Kk, 04.50.+h

Key words: Fermionic zero modes, large Extra Dimensions, self-dual vortex

* The project supported by National Natural Science Foundation of China under Grant No. 10275030 and 10475034 and the Fundamental Research Fund for Physics and Mathematic of Lanzhou University (No. Lzu07002).

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I. INTRODUCTION

In four-dimensional space-time, the interactions of fermions in a Nielsen-Olesen vortex background have been widely analyzed in the literature, mainly in connection with bound states at threshold [1], zero modes [2] and scattering solutions [3]. In Ref. [4], Stojkovic *et al* discussed some interesting situations arising in cases when fermions can have a non-trivial mass matrix. Such situations can also arise in higher dimensional models and can change the conclusions of fermion localization on a vortex (existence and non-existence of fermionic zero modes). Recently, Frere, Libanov and Troitsky have shown that a single family of fermions in six dimensions with vector-like couplings to the Standard Model (SM) bosons gives rise to three generations of chiral SM fermions in four dimensions [5]. In 5+1 dimensions, Frere *et al* also studied the fermionic zero modes in the background of a vortex-like solution on an extra two-dimensional sphere and relate them to the replication of fermion families in the SM [6].

The topological vortex (especially Abrikosov-Nielsen-Olesen vortex) coupled to fermions may lead to chiral fermionic zero modes [7]. Usually the number of the zero modes coincides with the topological number, that is, with the magnetic flux of the vortex. In Large Extra Dimensions (LED) models, the chiral fermions of the SM are described by the zero modes of multi-dimensional fermions localized in the (four-dimensional) core of a topological defect [8]. One spinor field of theory in 5+1 dimensions corresponds to three chiral fermions of effective theory in 3+1 dimensions. Due to this fact, number of free parameters of the model can be significantly reduced.

In Refs. [9, 10], we present a unified description of the topological and non-topological self-dual vortices on a two-dimensional non-compact extra space. Based on this vortex background, we study fermionic zero modes on the extra space in 5+1 dimensions [10]. Through two simple cases, it is shown that the vortex background contributes a phase shift to the fermionic zero modes. The phase is actually originated from the Aharonov-Bohm (AB) effect and can be divided into two parts, one is related with the topological number of the extra space, the other depends on the non-topological vortex solution. While the model of 5+1 dimensions with two extra compact dimensions can provide an interesting insight into the problem of hierarchy problem of chiral fermionic mass pattern. In this paper, we shall study fermionic zero modes coupled with a self-dual vortex background on two-dimensional compact extra spaces T^2 in 5+1 dimensions.

The paper is organized as follows: In section II, through the self-duality equation on

a two-dimensional curved space, we give the topological structure of self-dual vortex. In section III, we analyze the Dirac operator and the effective lagrangian of the fermions in the self-dual vortex background on a torus in 5+1 dimensions, and two simple cases are discussed to show the role of vortex background in the fermionic zero modes. In the last section, a brief conclusion is presented.

II. SELF-DUAL VORTEX ON A TWO-DIMENSIONAL CURVED RIEMANN SURFACE

We consider a (5+1)-dimensional space-time $M^4 \times K^2$ with M^4 represents our four-dimensional space-time and K^2 represents a two-dimensional extra Riemann surface. The metric G_{MN} of the manifold $M^4 \times K^2$ is determined by

$$ds^2 = G_{MN}dx^M dx^N = g_{\mu\nu}dx^\mu dx^\nu - \gamma_{ij}dy^i dy^j, \quad (1)$$

where $g_{\mu\nu} = g_{\mu\nu}(x)$ is the four-dimensional metric of the manifold M^4 , $\gamma_{ij} = \gamma_{ij}(y)$ is the two-dimensional metric of the extra space K^2 .

To generate the vortex solution, we introduce the Abelian Higgs Lagrangian

$$\mathcal{L}_{AH} = \sqrt{-G} \left(-\frac{1}{4}F_{MN}F^{MN} + (D^M \phi)^\dagger (D_M \phi) - \frac{\lambda}{2}(\|\phi\|^2 - v^2)^2 \right), \quad (2)$$

where $G = \det(G_{MN})$, $F_{MN} = \partial_M A_N - \partial_N A_M$, $\phi = \phi(y^k)$ is a complex scalar field on K^2 , $\|\phi\| = (\phi\phi^*)^{\frac{1}{2}}$, A_M is a U(1) gauge field, $D_M = \partial_M - ieA_M$ is gauge-covariant but not covariant under general coordinate transformation. The Abrikosov-Nielsen-Olesen vortex solution on the $M^4 \times K^2$ could be generated from the Higgs field. In the generalized Abelian Higgs model, if the system admits a Bogomol'nyi limit [11], one can arrive at the first-order Bogomol'nyi self-duality equations in a curved space-time [12]:

$$B = \mp e(\|\phi\|^2 - v^2), \quad (3)$$

$$D_i \phi \mp i\sqrt{\gamma}\epsilon_{ij}\gamma^{jk}D_k \phi = 0. \quad (4)$$

The complex Higgs field ϕ can be regarded as the complex representation of a two-dimensional vector field $\vec{\phi} = (\phi^1, \phi^2)$ over the base space, it is actually a section of a complex line bundle on the base manifold. Substituting $\phi = \phi^1 + i\phi^2$ and $D_i = \partial_i - ieA_i$ into Eq. (4) and splitting the real part from the imaginary part, we obtain two equations

$$\partial_i \phi^1 = -eA_i \phi^2 \mp \sqrt{\gamma}\epsilon_{ij}\gamma^{jk}(+\partial_k \phi^2 - eA_k \phi^1), \quad (5)$$

$$\partial_i \phi^2 = +eA_i \phi^1 \mp \sqrt{\gamma}\epsilon_{ij}\gamma^{jk}(-\partial_k \phi^1 - eA_k \phi^2). \quad (6)$$

From Eqs. (5) and (6), by calculating $\partial_i \phi^* \phi - \partial_i \phi \phi^*$, we can obtain the expression of the gauge potential

$$eA_i = -\frac{1}{2i\|\phi\|^2}(\partial_i \phi^* \phi - \partial_i \phi \phi^*) \mp \sqrt{\gamma} \epsilon_{ij} \gamma^{jk} \partial_k \ln \|\phi\|. \quad (7)$$

If we define the unit vector

$$n^a = \frac{\phi^a}{\|\phi\|}, \quad (a, b = 1, 2) \quad (8)$$

and note the identity

$$\epsilon_{ab} n^a \partial_i n^b = \frac{1}{2i\|\phi\|^2}(\partial_i \phi^* \phi - \partial_i \phi \phi^*), \quad (9)$$

Eq. (7) further simplifies to:

$$eA_i = -\epsilon_{ab} n^a \partial_i n^b \mp \sqrt{\gamma} \epsilon_{ij} \gamma^{jk} \partial_k \ln \|\phi\|. \quad (10)$$

In curved space, the magnetic field is defined by $B = -\frac{1}{\sqrt{\gamma}} \epsilon^{ij} \partial_i A_j$, according to Eq. (10), we have

$$e\sqrt{\gamma}B = \epsilon^{ij} \epsilon_{ab} \partial_i n^a \partial_j n^b \pm \epsilon^{ij} \epsilon_{jk} \partial_i (\sqrt{\gamma} \gamma^{kl} \partial_l \ln \|\phi\|). \quad (11)$$

So the first self-duality equation (3) can be generalized to

$$\mp e^2 \sqrt{\gamma} (\|\phi\|^2 - v^2) = \epsilon^{ij} \epsilon_{ab} \partial_i n^a \partial_j n^b \pm \epsilon^{ij} \epsilon_{jk} \partial_i (\sqrt{\gamma} \gamma^{kl} \partial_l \ln \|\phi\|). \quad (12)$$

According to Duan's ϕ -mapping topological current theory [13], it is easy to see that the first term on the RHS of Eq. (12) bears a topological origin, and the topological term just describes the non-trivial distribution of \vec{n} at large distances in space [14]. Noticing $\partial_i n^a = \partial_i \phi^a / \|\phi\| + \phi^a \partial_i (1/\|\phi\|)$ and the Green function relation in ϕ -space: $\partial_a \partial_a \ln(\|\phi\|) = 2\pi \delta^2(\vec{\phi})$, ($\partial_a = \frac{\partial}{\partial \phi^a}$), it can be proved that [15]

$$\epsilon^{ij} \epsilon_{ab} \partial_i n^a \partial_j n^b = 2\pi \delta^2(\vec{\phi}) J\left(\frac{\phi}{y}\right) = 2\pi \sum_{k=1}^N W_k \delta(\vec{y} - \vec{y}_k), \quad (13)$$

where $J(\phi/y)$ is the Jacobian and $W_k = \beta_k \eta_k$ is the winding number around the k -th vortex, the positive integer β_k is the Hopf index and $\eta_k = \pm 1$ is the Brouwer degree, \vec{y}_k are the coordinates of the k -th vortex. So the first Bogomol'nyi self-duality equation (3) should be

$$\mp e^2 \sqrt{\gamma} (\|\phi\|^2 - v^2) = 2\pi \sum_{k=1}^N W_k \delta(\vec{y} - \vec{y}_k) \mp \epsilon^{ij} \epsilon_{jk} \partial_i (\sqrt{\gamma} \gamma^{kl} \partial_l \ln \|\phi\|). \quad (14)$$

Obviously the first term on the RHS of Eq. (14) describes the topological self-dual vortex.

Now let us discuss the case of flat space for the self-duality equation (14). In this special case, $\gamma_{ij} = \delta_{ij}$ and Eq. (14) reads as

$$\mp e^2(\|\phi\|^2 - v^2) = 2\pi \sum_{k=1}^N W_k \delta(\vec{y} - \vec{y}_k) \mp \partial_i \partial_i \ln \|\phi\|. \quad (15)$$

While the corresponding conventional self-duality equation is [16]

$$e^2(\|\phi\|^2 - v^2) = \partial_i \partial_i \ln \|\phi\|. \quad (16)$$

Comparing our equation (15) with Eq. (16), one can see that the topological term $2\pi \sum_{k=1}^N W_k \delta(\vec{y} - \vec{y}_k)$, which describes the topological self-dual vortex, is missed in the conventional equation. Obviously, only when the field $\phi \neq 0$, the topological term vanishes and the conventional equation is correct. So, the exact self-duality equation should be Eq. (15) for flat space and Eq. (14) for curved one. As for conventional self-dual nonlinear equation (16), a great deal of work has been done by many physicists on it, and a vortex-like solution was given by Jaffe [17]. But no exact solutions are known.

For the case of a flat torus, $\gamma_{ij} dy^i dy^j = R_1^2 d\theta^2 + R_2^2 d\varphi^2$, the corresponding self-dual vortex equation is

$$\mp e^2 R_1 R_2 (\|\phi\|^2 - v^2) = 2\pi \sum_{k=1}^N W_k \delta(\vec{y} - \vec{y}_k) \mp \left(\frac{R_2}{R_1} \partial_\theta^2 \ln \|\phi\| + \frac{R_1}{R_2} \partial_\varphi^2 \ln \|\phi\| \right). \quad (17)$$

For the case of a sphere, $\gamma_{ij} dy^i dy^j = R^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$, and we have

$$\mp e^2 R^2 \sin \theta (\|\phi\|^2 - v^2) = 2\pi \sum_{k=1}^N W_k \delta(\vec{y} - \vec{y}_k) \mp \left(\partial_\theta (\sin \theta \partial_\theta \ln \|\phi\|) + \frac{1}{\sin \theta} \partial_\varphi^2 \ln \|\phi\| \right). \quad (18)$$

We shall study fermionic zero modes coupled with the vortex background on a torus in the following section.

III. FERMIONIC ZERO MODES IN VORTEX BACKGROUND ON A TORUS

The lagrangian of the fermions in the vortex background (10) on a torus is

$$\mathcal{L} = \sqrt{-G} \bar{\Psi} \{ i \Gamma^A E_A^M (\partial_M - \Omega_M - i e A_M) - g \phi \} \Psi, \quad (19)$$

where E_A^M is the *sechsbein* with

$$E_M^A = (e_\mu^a \delta_a^A, R_1 \delta_4^A, R_2 \delta_5^A). \quad (20)$$

The components of Ω_M are

$$\Omega_\mu = \omega_\mu, \quad \Omega_4 = 0, \quad \Omega_5 = 0, \quad (21)$$

where $\omega_\mu = \frac{1}{2}\omega_\mu^{ab}I_{ab}$ is the spin connection derived from the metric $g_{\mu\nu}(x) = e_\mu^a e_\nu^b \eta_{ab}$, lower case Latin indices $a, b = 0, \dots, 3$ correspond to the flat tangent four-dimensional Minkowski space.

Using Eq. (21), the lagrangian (19) of the fermions then becomes

$$\begin{aligned} \mathcal{L} = \sqrt{-G}\bar{\Psi} \left\{ i\Gamma^a e_a^\mu (\partial_\mu - \omega_\mu - ieA_\mu) + i\frac{\Gamma^4}{R_1}(\partial_\theta - ieA_\theta) \right. \\ \left. + i\frac{\Gamma^5}{R_2}(\partial_\varphi - ieA_\varphi) - g\phi \right\} \Psi. \end{aligned} \quad (22)$$

We apply now the standard decomposition procedure. Since the vortex background does not depend on x^μ , one can separate variables related to M^4 and T^2 . First, let us introduce the transverse Dirac operator D_T on a torus in the background (10):

$$D_T = i\frac{\bar{\Gamma}\Gamma^4}{R_1}(\partial_\theta - ieA_\theta) + i\frac{\bar{\Gamma}\Gamma^5}{R_2}(\partial_\varphi - ieA_\varphi) - \bar{\Gamma}g\phi, \quad (23)$$

where

$$\bar{\Gamma} = \Gamma^0\Gamma^1\Gamma^2\Gamma^3 = \begin{pmatrix} i\gamma^5 & 0 \\ 0 & -i\gamma^5 \end{pmatrix}. \quad (24)$$

Then expand any spinor in a set of eigenvectors $\Theta_m(\theta, \varphi)$ of this operator D_T

$$D_T\Theta_m(\theta, \varphi) = \lambda_m\Theta_m(\theta, \varphi). \quad (25)$$

For the zero modes of D_T , we have

$$D_T\Theta(\theta, \varphi) = 0. \quad (26)$$

This is just the Dirac equation on a torus with vortex backgrounds. For fermionic zero modes, we can write

$$\Psi(x, \theta, \varphi) = \psi(x)\Theta(\theta, \varphi), \quad (27)$$

where Θ satisfies Eq. (26). The effective Lagrangian for ψ then becomes

$$\mathcal{L}_{eff} = \int d\theta d\varphi \mathcal{L} = \mathcal{L}_\psi \int d\theta d\varphi R_1 R_2 \Theta^\dagger \Theta, \quad (28)$$

where

$$\mathcal{L}_\psi = \sqrt{-\det(g_{\mu\nu})} i\bar{\psi}\Gamma^a e_a^\mu (\partial_\mu - \omega_\mu - ieA_\mu)\psi. \quad (29)$$

Thus, to have the finite kinetic energy for ψ , the above integral must be finite. This can be achieved if the function $\Theta(\theta, \varphi)$ does not diverge on the whole torus.

In what follows, to illustrate how the vortex background affects the fermionic zero modes, we first discuss the simple case that the Higgs field ϕ is only relative to φ , and then solve the general Dirac equation for the vacuum Higgs field solution $\|\phi\| = v$.

A. Case I: ϕ is only relative to φ

Now we discuss a simple situation that ϕ only depends on the parameter φ , i.e., $\phi = \phi(\varphi)$. In this case, Eq. (10) reduces to:

$$A_\theta = \mp \frac{1}{e} \frac{R_1}{R_2} \partial_\varphi \ln \|\phi\|, \quad (30)$$

$$A_\varphi = -\frac{1}{e} \epsilon_{ab} n^a \partial_\varphi n^b, \quad (31)$$

and Dirac equation $D_T \Theta = 0$ becomes:

$$i\bar{\Gamma} \left\{ \frac{\Gamma^4}{R_1} \partial_\theta + \frac{\Gamma^5}{R_2} \left(\partial_\varphi - ie \frac{R_2}{R_1} A_\theta \Gamma^4 \Gamma^5 - ie A_\varphi \right) + ig \phi \right\} \Theta(\theta, \varphi) = 0. \quad (32)$$

Here Θ can be written as the following form:

$$\Theta(\theta, \varphi) = \exp \left[i \left(n + \frac{1}{2} \right) \theta \right] h(\varphi), \quad (33)$$

where n is an integer, and $h(\varphi)$ satisfies

$$\left\{ \partial_\varphi + i \frac{R_2}{R_1} \left(n + \frac{1}{2} - e A_\theta \right) \Gamma^4 \Gamma^5 - ie A_\varphi - ig R_2 \phi \Gamma^5 \right\} h(\varphi) = 0. \quad (34)$$

Substituting Eq. (30) into Eq. (34), we get

$$\left\{ \partial_\varphi - \left[\frac{R_2}{R_1} \left(n + \frac{1}{2} \right) \pm \partial_\varphi \ln \|\phi\| \right] \begin{pmatrix} \gamma^5 & 0 \\ 0 & \gamma^5 \end{pmatrix} - ie A_\varphi - ig R_2 \phi \begin{pmatrix} 0 & \gamma^0 \\ -\gamma^0 & 0 \end{pmatrix} \right\} h(\varphi) = 0 \quad (35)$$

with

$$h(\varphi) = \begin{pmatrix} h_1(\varphi) \\ h_2(\varphi) \end{pmatrix}. \quad (36)$$

It is very difficult to solve explicitly the above Dirac equation. Let us consider the case of small extra dimensions, i.e., $R_1, R_2 \ll 1$. Under this assumption, we can ignore the last item

on the LHS of Eq. (35). And imposing the chirality condition $\gamma^5 h_i(\varphi) = +h_i(\varphi)$, ($i = 1, 2$), Eq. (35) can be simplified as

$$\left\{ \partial_\varphi - \left[\frac{R_2}{R_1} \left(n + \frac{1}{2} \right) \pm \partial_\varphi \ln \|\phi\| + ieA_\varphi \right] \right\} h_i(\varphi) = 0. \quad (37)$$

So the fermionic zero mode is

$$\begin{aligned} \Theta(\theta, \varphi) = & \|\phi\|^{\pm 1} \exp \left[i \left(n + \frac{1}{2} \right) \theta + \frac{R_2}{R_1} \left(n + \frac{1}{2} \right) \varphi \right. \\ & \left. + ie \int d\varphi A_\varphi \right] \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \end{aligned} \quad (38)$$

Now we see, the gauge potential A_φ appears as phase factor to the fermionic zero modes. So the phase factor is decided by the gauge potential A_φ , but has no relation with A_θ . And A_φ has no contribution to effective Lagrangian, but A_θ does. Furthermore, for the anti-vortex, the corresponding zero mode is singular at the zero of ϕ . It suggests that the symmetric phase just corresponds to singularity of fermionic zero mode, the topological self-dual vortex just arises from the symmetric phase of Higgs field. So this singularity must bear a topological origin, it is determined by the topology of the physical system.

As all known, quantum topological and geometrical phases are ubiquitous in modern physics—in cosmology, particle physics, modern string theory and condensed matter. In fact, according to Eq. (38), we see this phase shift is actually the quantum mechanical AB phase. This discussion can be generalized to the AB phase of non-Abelian gauge theories, such as the Wilson and 't Hooft loops. Since the AB phase is fundamental to theories of anyons and to gauge fields, it is an important tool for studying the issues of confinement and spontaneous symmetry breaking.

B. Case II: the vacuum solution

In this subsection, we choose polar coordinates (r, χ) which origin is at the center of vortex, and discuss the fermions around the vortex for the case of the vacuum solution. The relation of (r, χ) and (θ, φ) is

$$R_1 \theta = y^1 = r \cos \chi, \quad R_2 \varphi = y^2 = r \sin \chi. \quad (39)$$

In the case, the gauge is decided by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu - dr^2 - r^2 d\chi^2. \quad (40)$$

Sechsbein E_M^A and the non-zero components of Ω_M are

$$E_M^A = (e_\mu^a \delta_a^A, \delta_4^A, r \delta_5^A), \quad (41)$$

$$\Omega_\mu = \omega_\mu, \quad \Omega_5 = \frac{1}{2} \Gamma^4 \Gamma^5, \quad (42)$$

For the vacuum solution, $\|\phi\|^2 = v^2$, i.e. $\phi = v e^{i\chi}$, $n^r = \cos \chi$, $n^\chi = \sin \chi$, we have $eA_r = 0$, $eA_\chi = -1$. Dirac equation $D_T \Theta = 0$ is

$$i\bar{\Gamma} \left\{ \Gamma^4 r \partial_r + \Gamma^5 \left(\partial_\chi - \frac{1}{2} \Gamma^4 \Gamma^5 + i \right) + i g r v e^{i\chi} \right\} \Theta(r, \chi) = 0. \quad (43)$$

Considering $r \ll \sqrt{R_1^2 + R_2^2} \ll 1$, we can ignore again the last item on the RHS of Eq. (43) and have $\Theta = f(r)h(\chi) = C h(\chi)$, where $h(\chi) = (h_1(\chi), h_2(\chi))^T$ satisfies the following equation

$$\left\{ \partial_\chi - \frac{i}{2} \begin{pmatrix} \gamma^5 & 0 \\ 0 & \gamma^5 \end{pmatrix} + i \right\} \begin{pmatrix} h_1(\chi) \\ h_2(\chi) \end{pmatrix} = 0. \quad (44)$$

Imposing the chirality condition $\gamma^5 h_i(\chi) = +h_i(\chi)$, we get the following solution:

$$\Theta(r, \chi) = C e^{-\frac{i}{2}\chi} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad (45)$$

So the fermionic zero mode around the vortex does not depend on the coupling constant g between extra space and Higgs field and the vacuum expectation value v .

Now we come to the issue of the presence of the zero modes. The number of zero-modes of the Dirac operator is decided by the index of it. The index of the Dirac operator on manifold K^2 is defined as the difference $n_+ - n_-$ between the number n_+ of right-handed four-dimensional fermions obtained by dimensional reduction and the number n_- of left-handed 4D fermions. This number is a topological quantity of the manifold upon compactification and the gauge bundles the Dirac operator might be coupled to. Indeed, this index can be computed in terms of characteristic classes of the tangent and gauge bundles. The Atiyah–Singer index theorem in two dimensions gives the difference [18]

$$n_+ - n_- = \frac{e}{4\pi} \int_{K^2} d^2 q \varepsilon^{ij} F_{ij}, \quad (46)$$

where F_{ij} is the field strength of A_i ,

$$F_{ij} \equiv \partial_i A_j - \partial_j A_i - ie[A_i, A_j]. \quad (47)$$

If we take $K^2 = S^2$ with a U(1) magnetic monopole field of charge n on it, the number of chiral families will then be equal to n [19]. We can also consider zero modes of the Dirac operator in the background of Abelian gauge potentials representing Dirac strings and center vortices on the torus T^2 . The result is for a two-vortex gauge potential (smeared out vortices) there is one normalizable zero mode which has exactly one zero on the torus [20]. The probability density of the spinor field is peaked at the positions of the vortices.

IV. SUMMARY AND DISCUSSIONS

Using the generalized Abelian Higgs model and ϕ -mapping theory, we investigate the self-dual vortex on an extra two-dimensional curved Riemann surface, and obtain the inner topological structure of the self-dual vortex. We also establish the exact self-duality equation with topological term, which is the density of topological charge of the vortex: $J^0 = \frac{1}{\sqrt{\gamma}} \frac{2\pi}{e} \delta^2(\vec{\phi}) J(\phi/y)$. Different from the conventional self-duality equation, our equation with topological term is valid even at the zero points of the field. In Abelian Higgs model, there are two kinds of vortex: the topological and non-topological self-dual vortices, which just arise from the symmetric and asymmetric phase of the Higgs field, respectively, and are described by a unified equation: $e\sqrt{\gamma}B = 2\pi \sum_{k=1}^N W_k \delta(\vec{y} - \vec{y}_k) \pm \epsilon^{ij} \epsilon_{jk} \partial_i (\sqrt{\gamma} \gamma^{kl} \partial_l \ln \|\phi\|)$. As two instances, we give the self-duality vortex equations on a torus and a sphere.

We give the relation between gauge potential and Higgs field. Based on this relation, the fermionic zero modes in self-dual vortex background on a torus are studied. When Higgs field is only relative to φ , the gauge potential A_φ appears as phase factor to the fermionic zero modes. So the phase factor is decided by the gauge potential A_φ . Furthermore, for the anti-vortex, the corresponding zero mode is singular at the zero of ϕ . It suggests that the symmetric phase just corresponds to singularity of fermionic zero mode, the topological self-dual vortex just arises from the symmetric phase of Higgs field. For the vacuum solution, the fermionic zero modes do not depend on the coupling constant g between extra space and Higgs field and the vacuum expectation value v .

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